

On potentially $K_{r+1} - U$ -graphical Sequences *

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Abstract

Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). We use the symbol Z_4 to denote $K_4 - P_2$. A sequence S is potentially $K_m - H$ -graphical if it has a realization containing a $K_m - H$ as a subgraph. Let $\sigma(K_m - H, n)$ denote the smallest degree sum such that every n -term graphical sequence S with $\sigma(S) \geq \sigma(K_m - H, n)$ is potentially $K_m - H$ -graphical. In this paper, we determine the values of $\sigma(K_{r+1} - U, n)$ for $n \geq 5r + 18$, $r + 1 \geq k \geq 7$, $j \geq 6$ where U is a graph on k vertices and j edges which contains a graph $K_3 \cup P_3$ but not contains a cycle on 4 vertices and not contains Z_4 .

Key words: graph; degree sequence; potentially $K_{r+1} - U$ -graphic sequence; potentially $K_{r+1} - K_3 \cup P_3$ -graphic sequence

AMS Subject Classifications: 05C07, 05C35

1 Introduction

The set of all non-increasing nonnegative integers sequence $\pi = (d_1, d_2, \dots, d_n)$ is denoted by NS_n . A sequence $\pi \in NS_n$ is said to be graphic if it is the

*Project Supported by NSF of Fujian(Z0511034), Fujian Provincial Training Foundation for "Bai-Quan-Wan Talents Engineering", Project of Fujian Education Department and Project of Zhangzhou Teachers College.

degree sequence of a simple graph G on n vertices, and such a graph G is called a realization of π . The set of all graphic sequences in NS_n is denoted by GS_n . A graphical sequence π is potentially H -graphical if there is a realization of π containing H as a subgraph, while π is forcibly H -graphical if every realization of π contains H as a subgraph. If π has a realization in which the $r + 1$ vertices of largest degree induce a clique, then π is said to be potentially A_{r+1} -graphic. Let $\sigma(\pi) = d_1 + d_2 + \dots + d_n$, and $[x]$ denote the largest integer less than or equal to x . If G and G_1 are graphs, then $G \cup G_1$ is the disjoint union of G and G_1 . If $G = G_1$, we abbreviate $G \cup G_1$ as $2G$. We denote $G + H$ as the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{xy : x \in V(G), y \in V(H)\}$. Let K_k , C_k , T_k , and P_k denote a complete graph on k vertices, a cycle on k vertices, a tree on $k + 1$ vertices, and a path on $k + 1$ vertices, respectively. Let $K_m - H$ be the graph obtained from K_m by removing the edges set $E(H)$ of the graph H (H is a subgraph of K_m). Let F_k denote the friendship graph on $2k + 1$ vertices, that is, the graph of k triangles intersecting in a single vertex. For $0 \leq r \leq t$, denote the generalized friendship graph on $kt - kr + r$ vertices by $F_{t,r,k}$, where $F_{t,r,k}$ is the graph of k copies of K_t meeting in a common r set. We use the symbol Z_4 to denote $K_4 - P_2$. We use the symbol $G[v_1, v_2, \dots, v_k]$ to denote the subgraph of G induced by vertex set $\{v_1, v_2, \dots, v_k\}$. We use the symbol $\epsilon(G)$ to denote the number of edges in graph G .

Given a graph H , what is the maximum number of edges of a graph with n vertices not containing H as a subgraph? This number is denoted $ex(n, H)$, and is known as the Turán number. Mantel [21] proved that $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$. This was rediscovered by Turán [22] as a special case of his results on $ex(n, K_k)$. In terms of graphic sequences, the number $2ex(n, H) + 2$ is the minimum even integer l such that every n -term graphical sequence π with $\sigma(\pi) \geq l$ is forcibly H -graphical. Here we consider the following variant: determine the minimum even integer l such that every n -term graphical sequence π with $\sigma(\pi) \geq l$ is potentially H -graphical. We denote this minimum l by $\sigma(H, n)$. Erdős, Jacobson and Lehel [3] showed that $\sigma(K_k, n) \geq (k - 2)(2n - k + 1) + 2$ and conjectured that the equality holds. They proved that if π does not contain zero terms, this conjecture is true for $k = 3$, $n \geq 6$. The conjecture is confirmed in [6], [15], [16], [17] and [18].

Gould, Jacobson and Lehel [6] also proved that $\sigma(pK_2, n) = (p - 1)(2n - 2) + 2$ for $p \geq 2$; $\sigma(C_4, n) = 2\lfloor \frac{3n-1}{2} \rfloor$ for $n \geq 4$. They also pointed out that it would be nice to see where in the range for $3n - 2$ to $4n - 4$, the value $\sigma(K_4 - e, n)$ lies. Luo [19] characterized the potentially C_k graphic sequence for $k = 3, 4, 5$. Luo and Warner [20] characterized the potentially K_4 -graphic sequences. Yin and Yin [32] characterize the potentially

$(K_5 - e)$ -positive graphic sequences and give two simple necessary and sufficient conditions for a positive graphic sequence π to be potentially K_5 -graphic. Moreover, they also give a simple necessary and sufficient condition for a positive graphic sequence π to be potentially K_6 -graphic. Ferrara, Gould and Schmitt [5] determined $\sigma(F_k, n)$ for n sufficiently large. Ferrara [4] determined $\sigma(F_{t,0,k}, n)$ for a sufficiently large choice of n and determined $\sigma(F_{t,t-2,k}, n)$ for a sufficiently large choice of n . Yin and Chen [23] determined $\sigma(F_{t,t-1,k}, n)$ for $n \geq 3t + 2k^2 + 3k - 6$. Yin, Chen and Schmitt [24] determined $\sigma(F_{t,r,k}, n)$ for $k \geq 2, t \geq 3, 1 \leq r \leq t - 2$ and n sufficiently large. Gould et al. [6] determined $\sigma(K_{2,2}, n)$ for $n \geq 4$. Yin et al. [27-30] determined $\sigma(K_{r,s}, n)$ for $s \geq r \geq 2$ and sufficiently large n . Lai [8] determined $\sigma(K_4 - e, n)$ for $n \geq 4$. Yin, Li and Mao[26] determined $\sigma(K_{r+1} - e, n)$ for $r \geq 3, r + 1 \leq n \leq 2r$ and $\sigma(K_5 - e, n)$ for $n \geq 5$. Yin and Li[25] gave a good method (Yin-Li method) of determining the values $\sigma(K_{r+1} - e, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$ (In fact, Yin and Li[25] also determining the values $\sigma(K_{r+1} - ke, n)$ for $r \geq 2$ and $n \geq 3r^2 - r - 1$). After reading[25], using Yin-Li method Yin [31] determined $\sigma(K_{r+1} - K_3, n)$ for $n \geq 3r + 5, r \geq 3$. Lai [9] determined $\sigma(K_5 - K_3, n)$, for $n \geq 5$. Lai [10,11] determined $\sigma(K_5 - C_4, n), \sigma(K_5 - P_3, n)$ and $\sigma(K_5 - P_4, n)$, for $n \geq 5$. Determining $\sigma(K_{r+1} - H, n)$, where H is a tree on 4 vertices is more useful than a cycle on 4 vertices (for example, $C_4 \not\subset C_i$, but $P_3 \subset C_i$ for $i \geq 5$). So, after reading[25] and [31], using Yin-Li method Lai and Hu[12] determined $\sigma(K_{r+1} - H, n)$ for $n \geq 4r + 10, r \geq 3, r + 1 \geq k \geq 4$ and H be a graph on k vertices which containing a tree on 4 vertices but not containing a cycle on 3 vertices and $\sigma(K_{r+1} - P_2, n)$ for $n \geq 4r + 8, r \geq 3$. Using Yin-Li method Lai and Sun[13] determined $\sigma(K_{r+1} - (kP_2 \cup tK_2), n)$ for $n \geq 4r + 10, r + 1 \geq 3k + 2t, k + t \geq 2, k \geq 1, t \geq 0$. To now, the problem of determining $\sigma(K_{r+1} - H, n)$ for H not containing a cycle on 3 vertices and sufficiently large n has been solved. Using Yin-Li method Lai[14] determine the values of $\sigma(K_{r+1} - Z, n)$ for $n \geq 5r + 19, r + 1 \geq k \geq 5, j \geq 5$ where Z is a graph on k vertices and j edges which contains a graph Z_4 but not contains a cycle on 4 vertices. Using Yin-Li method Lai[14] also determine the values of $\sigma(K_{r+1} - Z_4, n), \sigma(K_{r+1} - (K_4 - e), n), \sigma(K_{r+1} - K_4, n)$ for $n \geq 5r + 16, r \geq 4$. In this paper, using Yin-Li method we prove the following two theorems.

Theorem 1.1. If $r \geq 6$ and $n \geq 5r + 18$, then

$$\sigma(K_{r+1} - (K_3 \cup P_3), n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

Theorem 1.2. If $n \geq 5r + 18$, $r + 1 \geq k \geq 7$, and $j \geq 6$, then

$$\sigma(K_{r+1} - U, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \text{if } n-r \text{ is even} \end{cases}$$

where U is a graph on k vertices and j edges which contains a graph $(K_3 \cup P_3)$ but not contains a cycle on 4 vertices and not contains Z_4 .

There are a number of graphs on k vertices and j edges which contains a graph $(K_3 \cup P_3)$ but not contains a cycle on 4 vertices and not contains Z_4 . (for example, $C_3 \cup C_{i_1} \cup C_{i_2} \cup \dots \cup C_{i_p}$ ($i_j \neq 4, j = 2, 3, \dots, p, i_1 \geq 5$), $C_3 \cup P_{i_1} \cup P_{i_2} \cup \dots \cup P_{i_p}$ ($i_1 \geq 3$), $C_3 \cup P_{i_1} \cup C_{i_2} \cup \dots \cup C_{i_p}$ ($i_j \neq 4, j = 2, 3, \dots, p, i_1 \geq 3$), etc)

2 Preparations

In order to prove our main result, we need the following notations and results.

Let $\pi = (d_1, \dots, d_n) \in NS_n, 1 \leq k \leq n$. Let

$$\pi_k'' = \begin{cases} (d_1 - 1, \dots, d_{k-1} - 1, d_{k+1} - 1, \dots, d_{d_k+1} - 1, d_{d_k+2}, \dots, d_n), \\ \text{if } d_k \geq k, \\ (d_1 - 1, \dots, d_{d_k} - 1, d_{d_k+1}, \dots, d_{k-1}, d_{k+1}, \dots, d_n), \\ \text{if } d_k < k. \end{cases}$$

Denote $\pi_k' = (d_1', d_2', \dots, d_{n-1}')$, where $d_1' \geq d_2' \geq \dots \geq d_{n-1}'$ is a rearrangement of the $n-1$ terms of π_k'' . Then π_k' is called the residual sequence obtained by laying off d_k from π .

Theorem 2.1 [25] Let $n \geq r + 1$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r+1} \geq r$. If $d_i \geq 2r - i$ for $i = 1, 2, \dots, r - 1$, then π is potentially A_{r+1} -graphic.

Theorem 2.2 [25] Let $n \geq 2r + 2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r+1} \geq r$. If $d_{2r+2} \geq r - 1$, then π is potentially A_{r+1} -graphic.

Theorem 2.3 [25] Let $n \geq r + 1$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r+1} \geq r - 1$. If $d_i \geq 2r - i$ for $i = 1, 2, \dots, r - 1$, then π is potentially $K_{r+1} - e$ -graphic.

Theorem 2.4 [25] Let $n \geq 2r + 2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-1} \geq r$. If $d_{2r+2} \geq r - 1$, then π is potentially $K_{r+1} - e$ -graphic.

Theorem 2.5 [7] Let $\pi = (d_1, \dots, d_n) \in NS_n$ and $1 \leq k \leq n$. Then $\pi \in GS_n$ if and only if $\pi_k' \in GS_{n-1}$.

Theorem 2.6 [2] Let $\pi = (d_1, \dots, d_n) \in NS_n$ with even $\sigma(\pi)$. Then $\pi \in GS_n$ if and only if for any $t, 1 \leq t \leq n-1$,

$$\sum_{i=1}^t d_i \leq t(t-1) + \sum_{j=t+1}^n \min\{t, d_j\}.$$

Theorem 2.7 [6] If $\pi = (d_1, d_2, \dots, d_n)$ is a graphic sequence with a realization G containing H as a subgraph, then there exists a realization G' of π containing H as a subgraph so that the vertices of H have the largest degrees of π .

Lemma 2.1 [31] If $\pi = (d_1, d_2, \dots, d_n) \in NS_n$ is potentially $K_{r+1} - e$ -graphic, then there is a realization G of π containing $K_{r+1} - e$ with the $r+1$ vertices v_1, \dots, v_{r+1} such that $d_G(v_i) = d_i$ for $i = 1, 2, \dots, r+1$ and $e = v_r v_{r+1}$.

Lemma 2.2 [14] Let $n \geq 2r$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-1} \geq r$, $d_{r+1} \geq r-1$. If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-2$, then π is potentially $K_{r+1} - e$ -graphic.

Lemma 2.3 [14] Let $\pi = (d_1, \dots, d_n) \in GS_n$ and G be a realization of π . If $\epsilon(G[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 1$, then there is a realization H of π such that $d_H(v_i) = d_i$ for $i = 1, 2, \dots, r+1$ and $v_r v_{r+1} \notin E(H)$.

Lemma 2.4 [14] Let $n \geq 2r+2$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-4} \geq r$,

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r) - 2, & \text{if } n-r \text{ is even} \end{cases}$$

If $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - (P_2 \cup K_2)$ -graphic.

Lemma 2.5 [14] Let $n \geq 2r$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-2} \geq r+1$, $d_{r+1} \geq r$, $d_r - 1 \geq d_{r+1+2}$. If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$, then π is potentially A_{r+1} -graphic.

3 Proof of Main Results

Lemma 3.1 Let $n \geq 2r+2$, $r \geq 4$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with $d_{r-2} \geq r-1$ and $d_{r+1} \geq r-2$,

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$, then π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic.

Proof. We consider the following two cases.

Case 1: $d_{r+1} \geq r - 1$.

Subcase 1.1: $d_{r-1} \geq r + 1$.

If $d_{r-2} \geq r + 2$, then π is potentially $K_{r+1} - e$ -graphic by Theorem 2.3. Hence, π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic.

If $d_{r-2} = r + 1$.

If $d_{r+1} = r + 1$, then $d_{r-2} = d_{r-1} = d_r = d_{r+1} = r + 1$. Suppose π is not potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic. Let H be a realization of π , then $\epsilon(H[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$. Let $S = (d_1, d_2, \dots, d_{r-3}, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, d_{r+2}, \dots, d_n)$, then by Theorem 2.1, S is potentially A_{r+1} -graphic (Denote $S' = (d'_1, d'_2, \dots, d'_n)$, where $d'_1 \geq d'_2 \geq \dots \geq d'_n$ is a rearrangement of the n terms of S . Therefore $S' \in GS_n$ by Lemma 2.3. Then S' satisfies the conditions of Theorem 2.1). Therefore, there is a realization G of S with v_1, v_2, \dots, v_{r+1} ($d(v_i) = d_i, i = 1, 2, \dots, r - 1$, $d(v_r) = d_r + 1, d(v_{r+1}) = d_{r+1} + 1$), the $r + 1$ vertices of highest degree containing a K_{r+1} by Theorem 2.7. Hence, $G - v_{r+1}v_r$ is a realization of π . Thus, π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic, which is a contradiction.

If $d_{r+1} = r$ or $d_{r+1} = r - 1$, then $d_{r-1} - 1 \geq r \geq d_{r+2}$. The residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 = d_1 - 1 \geq 2(r - 1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r - 1) - [(r - 1) - 2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$, and $d'_{(r-1)+1} = d'_r \geq r - 2$. By Lemma 2.2, π'_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic by $\{d_1 - 1, \dots, d_{r-2} - 1, d_{r-1} - 1\} \subseteq \{d'_1, \dots, d'_r\}$ and Lemma 2.1.

Subcase 1.2: $d_{r-1} \leq r$.

If $d_{r-2} \geq r + 1$, then $d_{r-2} - 1 \geq d_{r-1}$. The residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 = d_1 - 1 \geq 2(r - 1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r - 1) - [(r - 1) - 2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$, and $d'_{(r-1)+1} = d'_r \geq r - 2$. By Lemma 2.2, π'_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic by $\{d_1 - 1, \dots, d_{r-2} - 1\} \subseteq \{d'_1, \dots, d'_r\}$ and Lemma 2.1.

If $d_{r-2} = r$.

If $d_{r+1} = r$, then $d_{r-2} = d_{r-1} = d_r = d_{r+1} = r$. Suppose π is not potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic. Let H be a realization of π , then $\epsilon(H[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$. Let $S = (d_1, d_2, \dots, d_{r-3}, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, d_{r+2}, \dots, d_n)$. Denote $S' = (d'_1, d'_2, \dots, d'_n)$, where $d'_1 \geq d'_2 \geq \dots \geq d'_n$ is a rearrangement of the n terms of S . Therefore $S' \in GS_n$ by Lemma 2.3. The residual sequence $S''_{r+1} = (d''_1, \dots, d''_{n-1})$ obtained by laying off $d'_{r+1} = d_{r-1} = r$ from S' satisfies: $d''_1 = d'_1 - 1 \geq 2(r - 1) - 1, \dots, d''_{(r-1)-2} = d''_{r-3} \geq d_{r-3} - 1 \geq 2(r - 1) - [(r - 1) - 2]$, $d''_{(r-1)-1} = d''_{r-2} \geq r$, and $d''_{(r-1)+1} = d''_r \geq r - 1$. By Theorem 2.1, S''_{r+1} is potentially $A_{(r-1)+1}$ -graphic. Therefore, π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic by

$\{d_1 - 1, \dots, d_{r-3} - 1, d_r, d_{r+1}\} \subseteq \{d'_1, \dots, d'_r\}$ and Theorem 2.7, which is a contradiction.

If $d_{r+1} = r - 1$ and $d_r = r$, then $d_{r-2} = d_{r-1} = d_r = r$. The residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 = d_1 - 1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$, and $d'_{(r-1)+1} = d'_r \geq r - 2$. By Lemma 2.2, π'_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic by $\{d_1 - 1, \dots, d_{r-2} - 1\} \subseteq \{d'_1, \dots, d'_r\}$, $d'_{r-2} = d_r, d'_{r-1} = d_{r-2} - 1, d'_r = d_{r-1} - 1$ and Lemma 2.1.

If $d_{r+1} = r - 1$ and $d_r = r - 1$, then $d_{r-2} - 1 \geq d_{r+2}$. The residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off d_{r+1} from π satisfies: $d'_1 = d_1 - 1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$, and $d'_{(r-1)+1} = d'_r \geq r - 2$. By Lemma 2.2, π'_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic by $\{d_1 - 1, \dots, d_{r-2} - 1\} \subseteq \{d'_1, \dots, d'_r\}$, $d'_{r-2} = d_{r-2} - 1$ and Lemma 2.1.

If $d_{r-2} = r - 1$, then $d_{r-2} = d_{r-1} = d_r = d_{r+1} = r - 1$. Suppose π is not potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic. Let H be a realization of π , then $\epsilon(H[v_1, v_2, \dots, v_{r+1}]) \leq \epsilon(K_{r+1}) - 3$. Let $S = (d_1, d_2, \dots, d_{r-3}, d_{r-2}, d_{r-1}, d_r + 1, d_{r+1} + 1, d_{r+2}, \dots, d_n)$. Denote $S' = (d'_1, d'_2, \dots, d'_n)$, where $d'_1 \geq d'_2 \geq \dots \geq d'_n$ is a rearrangement of the n terms of S . Therefore $S' \in GS_n$ by Lemma 2.3. The residual sequence $S''_{r+1} = (d''_1, \dots, d''_{n-1})$ obtained by laying off $d'_{r+1} = d_{r-1} = r - 1$ from S' satisfies: $d''_1 = d'_1 - 1 \geq 2(r-1) - 1, \dots, d''_{(r-1)-2} = d''_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d''_{(r-1)-1} = d''_{r-2} = r - 1$, and $d''_{(r-1)+1} = d''_r = r - 1$. By Lemma 2.2, S''_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic by $\{d_1 - 1, \dots, d_{r-3} - 1, d_r, d_{r+1}, d_{r-2}\} = \{d''_1, \dots, d''_r\}$ and Lemma 2.1, which is a contradiction.

Case 2: $d_{r+1} = r - 2$.

Subcase 2.1: $d_{r-1} < d_{r-2}$.

If $d_{r-2} \geq r$, then the residual sequence $\pi'_{r+1} = (d'_1, \dots, d'_{n-1})$ obtained by laying off $d_{r+1} = r - 2$ from π satisfies: (1) $d'_i = d_i - 1$ for $i = 1, 2, \dots, r - 2$, (2) $d'_1 = d_1 - 1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r - 1$, and $d'_{(r-1)+1} = d'_r = d_r \geq r - 2$. By Lemma 2.2, π'_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic by $\{d_1 - 1, \dots, d_{r-2} - 1, d_{r-1}, d_r\} = \{d'_1, \dots, d'_r\}$ and Lemma 2.1.

If $d_{r-2} = r - 1$, then

$$\begin{aligned}
\sigma(\pi) &\leq (r-3)(n-1) + r-1 + (r-2)(n-r+2) \\
&= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+2) \\
&= (r-1)(2n-r) - 3(n-r) - 2 \\
&< \sigma(\pi),
\end{aligned}$$

which is a contradiction.

Subcase 2.2: $d_{r-1} = d_{r-2}$, then π'_{r+1} satisfies: $d'_1 \geq d_1 - 1 \geq 2(r-1) - 1, \dots, d'_{(r-1)-2} = d'_{r-3} \geq d_{r-3} - 1 \geq 2(r-1) - [(r-1) - 2]$, $d'_{(r-1)-1} = d'_{r-2} \geq r-1$ and $d'_{(r-1)+1} = d'_r \geq r-2$. By Lemma 2.2, π'_{r+1} is potentially $K_{(r-1)+1} - e$ -graphic. Therefore, π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic by $\{d_{r-1}, d_r, d_1 - 1, \dots, d_{r-2} - 1\} = \{d'_1, \dots, d'_r\}$ and Lemma 2.1.

Lemma 3.2. If $n \geq r+1, r+1 \geq k \geq 7$, then

$$\sigma(K_{r+1} - U, n) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

where U is a graph on k vertices and j edges which not contains a cycle on 4 vertices and not contains Z_4 .

Proof. Let

$$G = \begin{cases} K_{r-3} + (\frac{n-r-1}{2}K_2 \cup P_2 \cup K_1), & \text{if } n-r \text{ is odd} \\ K_{r-3} + (\frac{n-r}{2}K_2 \cup P_2), & \text{if } n-r \text{ is even} \end{cases}$$

Then G is a unique realization of

$$\pi = \begin{cases} ((n-1)^{r-3}, (r-1)^1, (r-2)^{n-r+1}, (r-3)^1) & \text{if } n-r \text{ is odd} \\ ((n-1)^{r-3}, (r-1)^1, (r-2)^{n-r+2}) & \text{if } n-r \text{ is even} \end{cases}$$

and G clearly does not contain $K_{r+1} - U$, where the symbol x^y means x repeats y times in the sequence. Thus $\sigma(K_{r+1} - U, n) \geq \sigma(\pi) + 2$. Therefore,

$$\sigma(K_{r+1} - U, n) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, & \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), & \text{if } n-r \text{ is even} \end{cases}$$

The Proof of Theorem 1.1 According to Lemma 3.2, it is enough to verify that for $n \geq 5r + 18$,

$$\sigma(K_{r+1} - (K_3 \cup P_3), n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \text{if } n-r \text{ is even} \end{cases}$$

We now prove that if $n \geq 5r + 18$ and $\pi = (d_1, d_2, \dots, d_n) \in GS_n$ with

$$\sigma(\pi) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \text{if } n-r \text{ is even} \end{cases}$$

then π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic.

If $d_{r-4} \leq r-1$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-5)(n-1) + (r-1)(n-r+5) \\ &= (r-1)(n-1) - 4(n-1) + (r-1)(n-r+5) \\ &= (r-1)(2n-r) - 4(n-r) \\ &< (r-1)(2n-r) - 3(n-r) - 1, \end{aligned}$$

which is a contradiction. Thus, $d_{r-4} \geq r$.

If $d_{r-2} \leq r-2$, then

$$\begin{aligned} \sigma(\pi) &\leq (r-3)(n-1) + (r-2)(n-r+3) \\ &= (r-1)(n-1) - 2(n-1) + (r-1)(n-r+3) - (n-r+3) \\ &= (r-1)(2n-r) - 3(n-r) - 3 \\ &< (r-1)(2n-r) - 3(n-r) - 1, \end{aligned}$$

which is a contradiction. Thus, $d_{r-2} \geq r-1$.

If $d_{r+1} \leq r-3$, then

$$\begin{aligned} \sigma(\pi) &= \sum_{i=1}^r d_i + \sum_{i=r+1}^n d_i \\ &\leq (r-1)r + \sum_{i=r+1}^n \min\{r, d_i\} + \sum_{i=r+1}^n d_i \\ &= (r-1)r + 2 \sum_{i=r+1}^n d_i \\ &\leq (r-1)r + 2(n-r)(r-3) \\ &= (r-1)(2n-r) - 4(n-r) \\ &< (r-1)(2n-r) - 3(n-r) - 1, \end{aligned}$$

which is a contradiction. Thus, $d_{r+1} \geq r-2$.

If $d_i \geq 2r-i$ for $i = 1, 2, \dots, r-3$ or $d_{2r+2} \geq r-1$, then π is potentially $K_{r+1} - (K_3 \cup P_3)$ -graphic by Lemma 3.1 or Lemma 2.4. If $d_{2r+2} \leq r-2$

and there exists an integer i , $1 \leq i \leq r-3$ such that $d_i \leq 2r-i-1$, then

$$\begin{aligned}\sigma(\pi) &\leq (i-1)(n-1) + (2r+1-i+1)(2r-i-1) \\ &\quad + (r-2)(n+1-2r-2) \\ &= i^2 + i(n-4r-2) - (n-1) \\ &\quad + (2r-1)(2r+2) + (r-2)(n-2r-1).\end{aligned}$$

Since $n \geq 5r+18$, it is easy to see that $i^2 + i(n-4r-2)$, consider as a function of i , attains its maximum value when $i = r-3$. Therefore,

$$\begin{aligned}\sigma(\pi) &\leq (r-3)^2 + (n-4r-2)(r-3) - (n-1) \\ &\quad + (2r-1)(2r+2) + (r-2)(n-2r-1) \\ &= (r-1)(2n-r) - 3(n-r) - n + 5r + 16 \\ &< \sigma(\pi),\end{aligned}$$

which is a contradiction.

Thus,

$$\sigma(K_{r+1} - (K_3 \cup P_3), n) \leq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \text{if } n-r \text{ is even} \end{cases}$$

for $n \geq 5r+18$.

The Proof of Theorem 1.2 By Lemma 3.2, for $n \geq 5r+18, r+1 \geq k \geq 7$, and $j \geq 6$,

$$\sigma(K_{r+1} - U, n) \geq \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \text{if } n-r \text{ is even} \end{cases}$$

Obviously, $\sigma(K_{r+1} - U, n) \leq \sigma(K_{r+1} - (K_3 \cup P_3), n)$. By theorem 1.1,

$$\sigma(K_{r+1} - (K_3 \cup P_3), n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \text{if } n-r \text{ is even} \end{cases}$$

Then

$$\sigma(K_{r+1} - U, n) = \begin{cases} (r-1)(2n-r) - 3(n-r) - 1, \\ \text{if } n-r \text{ is odd} \\ (r-1)(2n-r) - 3(n-r), \\ \text{if } n-r \text{ is even} \end{cases}$$

for $n \geq 5r+18, r+1 \geq k \geq 7$, and $j \geq 6$.

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